Problem set 06.

Exercise 1. There are 100 questions for the exam. The student knows the answers to 80 of them. The examiner stops the exam if the student is unable to answer a question, but no later than after the fifth question. The final grade is equal to the number of questions answered by the student. What is the probability that the student will receive at least a grade 4?

Exercise 2. We toss a balanced coin four times. Let X be a number of heads obtained. Determine the distribution of this random variable, its cumulative distribution function (draw its graph), and its expected value.

Exercise 3. In a factory, three workers deliver the same parts, with 30% of the parts coming from worker I, 45% from worker II, and the rest from worker III. The workers produce 2%, 1%, and 3% defects, respectively. What is the probability that a randomly selected part will turn out to be good?

Exercise 4. Consider families with three children and assume that each child is equally likely to be a boy or a girl. If such a family is picked at random and the eldest child is found to be a boy, what is the probability that the other two are girls? The same question if a randomly chosen child from the family turns out to be a boy.

Exercise 5. All the screws in a machine come from the same factory but it is as likely to be from Factory A as from Factory B. The percentage of defective screws is 5% from A and 1% from B. Two screws are inspected; if the first is found to be good, what is the probability that the second is also good?

Exercise 6. Based on life tables, we know that 89.835% of women live to age 60, and 57.062% to age 80. What is the probability that Mrs. Ala, who is 60, will live another twenty years?

Exercise 7. From a batch of five items, one item was randomly selected and turned out to be defective. Any number of defective items of $0, 1, 2, 3, 4$ or 5 in a batch is equally probable. What is the probability that there were more than 3 defective items in the batch?

Exercise 8. Among the 100 coins there is one coin with two heads. One coin was drawn and then tossed 6 times, getting heads each time. What is the probability that the chosen coin had two heads?

Exercise 9. A breath analyzer, used by the police to test whether drivers exceed the legal limit set for the blood alcohol percentage while driving, is known to satisfy

$$\mathrm{P}(A|B) = \mathrm{P}(A^c|B^c) = p,$$

where A is the event "breath analyzer indicates that legal limit is exceeded" and B "driver's blood alcohol percentage exceeds legal limit." On Saturday night about 5% of the drivers are known to exceed the limit.
a. Describe in words the meaning of $\mathrm{P}(B^c|A)$.
b. Determine $\mathrm{P}(B^c|A)$ if $p = 0.95$.
c. How big should p be so that $\mathrm{P}(B|A) = 0.9$?

Exercise 10. A student takes a multiple-choice exam. Suppose for each question he either knows the answer or gambles and chooses an option at random. Further suppose that if he knows the answer, the probability of a correct answer is 1, and if he gambles this probability is 1/4. To pass, students need to answer at least 60% of the questions correctly. The student has "studied for a minimal pass," i.e., with probability 0.6 he knows the answer to a question. Given that he answers a question correctly, what is the probability that he actually knows the answer?

Exercise 11. A student is applying to Harvard and Dartmouth. He estimates that he has a probability of 0.5 of being accepted at Dartmouth and 0.3 of being accepted at Harvard. He further estimates the probability that he will be accepted by both is 0.2. What is the probability that he is accepted by Dartmouth if he is accepted by Harvard? Is the event "accepted at Harvard" independent of the event "accepted at Dartmouth"?

Exercise 12. The Yankees are playing the Dodgers in a world series. The Yankees win each game with probability 0.6. What is the probability that the Yankees win the series? (The series is won by the first team to win four games.)

Exercise 13. An urn contains 5 balls: 2 white and 3 black. Balls are drawn one by one without replacement. The drawing process stops as soon as all the black balls have been removed. Calculate the expected number of white balls remaining in the urn.

Exercise 14. An urn contains four white balls and four black balls. Balls are drawn from the urn one by one, without replacement, until a black ball is drawn for the first time. Determine the expected number of white balls drawn.

Exercise 15. An urn contains 20 balls, 10 of which are white and 10 black. We draw 18 balls at random without replacement. Let N be the number of white balls drawn. Calculate the variance of the random variable N.

Exercise 16. From an urn containing five white, three red, and two green balls, four balls are drawn with replacement. Determine the probability distribution of the number of white balls drawn, the cumulative distribution function of the corresponding random variable, and its expected value.

Exercise 17. An urn contains five white balls and ten black balls. Balls are drawn one by one, without replacement, until both colors are represented among the drawn balls. What is the expected number of black balls drawn?

Exercise 18. We roll two dice: one red and one white. Let the random variable be the difference between the outcomes on the red and white dice. Determine the distribution of this random variable, its cumulative distribution function, and its expected value.

Exercise 19. We toss a coin repeatedly until, for the first time, we obtain the same result in two consecutive tosses. Calculate the expected number of tosses performed.

Exercise 20. Greg and Johnny alternately flip a coin. Johnny wins when the (H, H, T) appears, and Greg wins when the (T, H, T) appears. What are the probabilities of winning for each boy?

Exercise 21. A doctor assumes that a patient has one of three diseases d_1, d_2, or d_3. Before any test, he assumes an equal probability for each disease. He carries out a test that will be positive with probability 0.8 if the patient has d_1, 0.6 if he has disease d_2, and 0.4 if he has disease d_3. Given that the outcome of the test was positive, what probabilities should the doctor now assign to the three possible diseases?